ON SYMMETRIC BASIC SEQUENCES IN LORENTZ SEQUENCE SPACES

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ABSTRACT

We examine the symmetric basic sequences in some classes of Banach spaces with symmetric bases. We show that the Lorentz sequence space $d(a,p)$ has a unique symmetric basis and every infinite dimensional subspace of $d(a,p)$ contains a subspace isomorphic to l^p. The symmetric basic sequences in $d(a,p)$ are identified and a necessary and sufficient condition for a Lorents sequence space with exactly two nonequivalent symmetric basic sequences in given. We conclude by exhibiting an example of a Lorentz sequence space having a subspace with symmetric basis which is not isomorphic either to a Lorentz sequence space or to an l^p-space.

Introduction

A basis $\{x_n\}$ of a Banach space X is called symmetric if every permutation $\{x_{\sigma(n)}\}$ of $\{x_n\}$ is a basis of X, equivalent to the basis $\{x_n\}$. In this paper we consider the problem of constructing symmetric basic sequences in some Banach spaces with symmetric bases.

Much of our work is done with the Lorentz sequence spaces $d(a,p)$. Let $1 \leqq p < +\infty$. For any $a = (a_1, a_2, \cdots) \in c_0 \setminus l^1$, $a_1 \geqq a_2 \geqq \cdots \geqq 0$, let $d(a,p) = \{x = (\alpha_1, \alpha_2, \cdots) \in c_0 : \sup_{\sigma\in\pi} \sum_{i=1}^{\infty} |\alpha_{\sigma(i)}|^p a_n < +\infty\}$ where π is the set of all permutations of the natural numbers. Then $d(a,p)$ with the norm $\| x \| = (\sup_{\sigma\in\pi} \sum_{n=1}^{\infty} |\alpha_{\sigma(n)}|^p a_n)^{1/p}$ for $x \in d(a,p)$ is a Banach space and the sequence of unit vectors $\{x_n\}$ is a symmetric basis of $d(a, p)$ [2,4]. For $p = 1$, these spaces have been studied by W. L. C. Sargent [10], D. J. H. Garling [2], W. Ruckle [9], and J. R. Calder and J. B. Hill [1]. For $1 < p < +\infty$, Garling [4]

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showed that $d(a,p)$ is a reflexive Banach space which, in general, is distinct from the l^p-spaces. See [1] for further references on other work on $d(a,p)$.

Another class of Banach spaces with symmetric basis is that of the Orlicz sequence spaces. J. Lindenstrauss and L. Tzafriri [6, 7] have shown that every Orlicz sequence space has a subspace isomorphic to some l^p. They have also shown that there are Orlicz sequence spaces which have at least two nonequivalent symmetric bases. We show that $d(a,p)$ has a unique symmetric basis for all a and p and that every infinite dimensional subspace X of $d(a,p)$ has a subspace isomorphic to l^p which can be chosen to be complemented in X if X has a symmetric basis. The Lorentz sequence spaces which have exactly two nonequivalent symmetric basic sequences are characterized. Finally, an example of a Lorentz sequence space having a subspace with symmetric basis which is isomorphic neither to l_p nor to any Lorentz sequence space is given.

We introduce a new type of block basic sequence of a symmetric basis which has the property that it always has a symmetric subsequence. In the spaces $d(a,p)$, these are the only symmetric block basic sequences of the unit vector basis $\{x_n\}$ of $d(a,p)$ which are not equivalent to the unit vector basis of l^p.

The notations and terminology in this paper are essentially those of I. Singer [12]. A sequence $\{x_n\}$ of a Banach space X is called a basis of X if every $x \in X$ has a unique expansion of the form $x = \sum_{i=1}^{\infty} \alpha_n x_n$. Let $1 \leqq p < +\infty$; a basis $\{x_n\}$ of X is called p-Hilbertian if $\sum_{n=1}^{\infty} \alpha_n x_n$ converges in X for every $\{\alpha_n\} \in l^p$. A basis $\{x_n\}$ is q-Besselian, $1 \leqq q < +\infty$, if $\sum_{n=1}^{\infty} \alpha_n x_n$ converges in X implies that $\{\alpha_n\} \in l^q$.

If $\{x_n\}$ is a basis of a Banach space X, a sequence $\{y_n\}$ in X is said to be a block basic sequence of $\{x_n\}$ if there is an increasing sequence of natural numbers $\{p_n\}$ such that $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$ for $n = 1, 2, \cdots$. A block basic sequence $\{y_n\}$ is said to be bounded if $0 < \inf_{1 \leqq n < +\infty} \| y_n \| \leqq \sup_{1 \leqq n < +\infty} \| y_n \| < +\infty$. We will denote by $[\{y_n\}]$ the closed linear span of the sequence $\{y_n\}$. If $\{x_n\}$ and $\{y_n\}$ are bases of X and Y, respectively, we say that $\{x_n\}$ dominates $\{y_n\}$, and write $\{x_n\} > \{y_n\}$, if $\sum_{n=1}^{\infty} \alpha_n x_n$ converges in X implies that $\sum_{n=1}^{\infty} \alpha_n y_n$ converges in Y. The basis $\{x_n\}$ is equivalent to the basis $\{y_n\}$, and we write $\{x_n\} \sim \{y_n\}$, if $\{x_n\} > \{y_n\}$ and $\{y_n\} > \{x_n\}$. It is clear that a basis $\{x_n\}$ is equivalent to the unit vector basis of l^p if and only if $\{x_n\}$ is p-Hilbertian and p-Besselian.

If $\{x_n\}$ and $\{y_n\}$ are symmetric bases, it is easy to show that $\{x_n\} \sim \{y_n\}$ if and only if for any sequence of scalars $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$, $\sum_{n=1}^{\infty} \alpha_n x_n$ converges in X if and only if $\sum_{n=1}^{\infty} \alpha_n y_n$ converges in Y. We also note that if

$$y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$$

for $n = 1,2,\cdots$, is a block basic sequence of a symmetric basis $\{x_n\}$, and for each n, σ_n is a permutation of $\{p_n + 1, p_n + 2, \cdots, p_{n+1}\}$, then $\{y_n\} \sim \{z_n\}$ where $z_n = \sum_{i=p_n+1}^{p_{n+1}} |\alpha_{\sigma(i)}| x_i$, $n = 1,2,\cdots$. Therefore, when working with block basic sequences $\{y_n\}$ of a symmetric basis $\{x_n\}$ we will always assume that $\alpha_{p_n+1} \geqq \alpha_{p_n+2} \geqq \cdots \geqq \alpha_{p_{n+1}} \geqq 0$ for $n = 1,2,\cdots$.

Let $\{x_n\}$ be a symmetric basis in a Banach space X. Define

$$|||x||| = \sup_{\sigma\in\pi} \sup_{\substack{|\beta_i|\leqq 1\\ 1\leqq n<+\infty}} \left\| \sum_{i=1}^{n} \beta_i f_i(x) x_{\sigma(i)} \right\|, \qquad x \in X,$$

where $\{f_n\}$ is the sequence of biorthogonal functionals of $\{x_n\}$ in X^*. Then the symmetric norm $|||x|||$, $x \in X$, is an equivalent norm on X. Throughout this paper, we shall assume that every Banach space with symmetric basis is equipped with the symmetric norm.

1. Preliminaries

In this section we state some simple and well-known facts on symmetric basic sequences in Banach spaces.

PROPOSITION 1. *Every symmetric basic sequence in a Banach space is either weakly convergent to zero or is equivalent to the unit vector basis of l^1.*

It is known that in the l^p spaces, $1 \leqq p < \infty$, all symmetric bases are equivalent [12, p. 573]. As a consequence of Proposition 1, we have

COROLLARY 1. *In the spaces $X = c_0$ or $l^p, 1 \leqq p < +\infty$, all symmetric basic sequences are equivalent.*

PROPOSITION 2. *Let X be a Banach space with a symmetric basis $\{x_n\}$. If every bounded block basic sequence of $\{x_n\}$ is symmetric, then $\{x_n\}$ is equivalent to the natural basis of c_0 or l^p for some p, $1 \leqq p < +\infty$.*

PROOF. Let $\{y_n\}$ be a bounded block basic sequence of $\{x_n\}$. Since $\{y_n\}$ is symmetric, $\{y_n\} \sim \{y_{2n}\}$. Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$z_n = \begin{cases} y_{2i} & \text{if } n = 2i, \quad i = 1,2,\cdots, \\ x_{n_i} & \text{if } n = 2i+1, \quad i = 1,2,\cdots, \end{cases}$$

is a bounded block basic sequence of $\{x_n\}$. Then, since $\{z_n\}$ is symmetric,

$\{x_n\} \sim \{x_{n_i}\} \sim \{z_n\} \sim \{y_{2n}\} \sim \{y_n\}$. Hence by a result of M. Zippin [13], $\{x_n\}$ is equivalent to the natural basis of c_0 or l^p, $1 \leqq p < +\infty$. Q.E.D.

PROPOSITION 3. *Let $\{x_n\}$ be a symmetric basis of a Banach space X. If $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$, $n = 1,2,\cdots$ is a bounded block basic sequence of $\{x_n\}$ and $\sup_{1 \leqq n < +\infty}(p_{n+1} - p_n) < +\infty$, then $\{y_n\}$ is equivalent to $\{x_n\}$.*

PROOF. We may assume that $\|x_n\| = \|y_n\| = 1$ for $n = 1,2,\cdots$. Suppose $\sum_{n=1}^{\infty} a_n x_n$ converges in X. Since $\{x_n\}$ is symmetric and $|\alpha_{p_n+i}| \leqq \|y_n\| \leqq 1$, $\sum_{n=1}^{\infty} |a_n \alpha_{p_n+i}| x_{p_n+i}$ converges in X for each $i = 1,2,\cdots,M$ where $M = \sup_{1 \leqq n < +\infty}(p_{n+1} - p_n)$. Since

$$\left\| \sum_{n=1}^{\infty} a_n y_n \right\| \leqq \left\| \sum_{n=1}^{\infty} \sum_{i=1}^{M} |a_n \alpha_{p_n+i}| x_{p_n+i} \right\|,$$

the series $\sum_{n=1}^{\infty} a_n y_n$ converges in X.

Conversely, if $\sum_{n=1}^{\infty} a_n y_n$ converges in X, note that for each $n = 1,2,\cdots$, there exists k_n such that $p_n + 1 \leqq k_n \leqq p_{n+1}$ and $|\alpha_{k_n}| \geqq 1/M > 0$. Hence, $\sum_{n=1}^{\infty} a_n \alpha_{k_n} x_{k_n}$ converges in X and so $\sum_{n=1}^{\infty} a_n x_n$ converges in X. Q.E.D.

PROPOSITION 4. *Let $\{x_n\}$ be a symmetric basis in a Banach space X. If $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$, $n = 1,2,\cdots$, is a bounded block basic sequence of $\{x_n\}$ such that $\inf_{1 \leqq n < +\infty} \sup_{p_n+1 \leqq i \leqq p_{n+1}} |\alpha_i| > 0$, then $\{y_n\}$ dominates $\{x_n\}$. However, in general, $\{y_n\}$ is not equivalent to $\{x_n\}$.*

PROOF. Since $\{x_n\}$ is symmetric, we may assume that there exist $\varepsilon > 0$ and $0 \leqq k_n \leqq p_{n+1} - p_n$ such that $\alpha_{p_n+k_n} \geqq \varepsilon$ for $n = 1,2,\cdots$. Suppose $\sum_{n=1}^{\infty} a_n y_n$ converges in X. Then

$$\left\| \sum_{i=1}^{n} a_i x_{p_i+k_i} \right\| \leqq \frac{1}{\varepsilon} \left\| \sum_{i=1}^{n} a_i \alpha_{p_i+k_i} x_{p_i+k_i} \right\| \leqq \frac{1}{\varepsilon} \left\| \sum_{i=1}^{n} a_i y_i \right\|.$$

Thus $\sum_{n=1}^{\infty} a_n x_{p_n+k_n}$ converges in X, so that $\sum_{n=1}^{\infty} a_n x_n$ converges in X.

Now, let $\{x_n\}$ be any nonshrinking symmetric basis which is not equivalent to the unit vector basis $\{e_n\}$ of l^1 (e.g., the unit vector basis of the space d [12, p. 361]). Since $\{x_n\}$ is nonshrinking, there is a bounded block basic sequence $z_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$ for $n = 1,2,\cdots$, which is of type l_+ [12, p. 369]. Hence $\{z_n\} \sim \{e_n\}$ and is a symmetric basic sequence. Let $y_n = x_{p_{2n}} + z_{2n}$ for $n = 1,2,\cdots$. Then $\{y_n\}$ is a bounded block basic sequence of $\{x_n\}$ and it is clear that $\{y_n\}$ satisfies the hypothesis of Proposition 4. However, $\{y_n\} \sim \{z_{2n}\} \sim \{e_n\}$, so $\{y_n\}$ is not equivalent to $\{x_n\}$. Q.E.D

2. The Lorentz sequence spaces $d(a,p)$

Let $1 \leqq p < +\infty$. For any sequence $a = (a_1, a_2, \cdots) \in c_0 \setminus l^1$, $a_1 \geqq a_2 \geqq \cdots \geqq 0$, in the Lorentz sequence space $d(a,p)$, the unit vector basis $\{x_n\}$ is symmetric [2, 4]. For any $x = (\alpha_1, \alpha_2, \cdots) \in d(a,p)$, let $\hat{x} = (\hat{\alpha}_1, \hat{\alpha}_2, \cdots)$ where $\{\hat{\alpha}_n\}$ is an enumeration of the nonzero elements of $\{\alpha_n\}$ such that $|\hat{\alpha}_1| \geqq |\hat{\alpha}_2| \geqq \cdots$. Then it can be proved that $\|x\| = (\sum_{n=1}^{\infty} |\hat{\alpha}_n|^p a_n)^{1/p}$. In the rest of the paper, we shall assume that $a = (a_1, a_2, \cdots) \in c_0 \setminus l^1$, $1 \geqq a_1 \geqq a_2 \geqq \cdots \geqq 0$ and $1 \leqq p < +\infty$. It is clear that the norm in $d(a,p)$ is a symmetric norm.

Proposition 5. *If $\{x_n\}$ is the unit vector basis of $d(a,p)$ then all bounded block basic sequences of $\{x_n\}$ are p-Hilbertian. In particular, all symmetric basic sequences in $d(a,p)$ are p-Hilbertian.*

Proof. Let $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$, $n = 1, 2, \cdots$, be a block basic sequence of $\{x_n\}$ such that $\|y_n\| = 1$, $n = 1, 2, \cdots$. For any nonnegative scalars $b_1, b_2, \cdots, b_n$

$$\left\| \sum_{i=1}^{n} b_i y_i \right\| = \left\| \sum_{i=1}^{n} \sum_{j=p_i+1}^{p_{i+1}} b_i^p |\alpha_j|^p a_{i,j} \right\|^{1/p}$$

where $\{a_{i,j}\}_{j=p_i+1 \cdots p_{i+1}, i=1,2 \cdots, n}$ is an enumeration of $\{a_1, a_2, \cdots, a_k\}$ for some k. For each $i = 1, 2, \cdots, n$, $\sum_{j=p_i+1}^{p_{i+1}} |\alpha_j|^p a_{i,j} \leqq \|y_i\|^p = 1$. Hence $\|\sum_{i=1}^{n} b_i y_i\| \leqq (\sum_{i=1}^{n} b_i^p)^{1/p}$ and $\{y_n\}$ is p-Hilbertian. Q.E.D.

Lemma 1. *Let $\{x_n\}$ be the unit vector basis of $d(a,p)$. If $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$, $n = 1, 2, \cdots$, is a bounded block basic sequence of $\{x_n\}$ such that $\lim_{n\to\infty} \alpha_n = 0$, then there exists a subsequence of $\{y_n\}$ which is equivalent to the unit vector basis of l^p.*

Proof. Since $\{x_n\}$ is a symmetric basis, and $\lim_{n\to\infty} \alpha_n = 0$, by switching to a subsequence if necessary, we may assume that $\alpha_{p_1+1} \geqq \alpha_{p_1+2} \geqq \cdots \geqq \alpha_n \geqq \cdots \geqq 0$, $p_{n+2} - p_{n+1} \geqq p_{n+1} - p_n$ and $\|y_n\| = 1$ for $n = 1, 2, \cdots$. We shall construct a block basic sequence $z_n = \sum_{i=q_n+1}^{q_{n+1}} \beta_i x_i$ for $n = 1, 2, \cdots$ of $\{x_n\}$ with the following two properties:

(1) $\|z_n\| = 1$ and $\sum_{i=q_n+1}^{q_{n+1}} \beta_i^p a_i \geqq \frac{1}{2}$ for $n = 1, 2, \cdots$;

(2) $\{z_n\}$ is equivalent to a subsequence of $\{y_n\}$.

We may assume that $p_1 = 1$ and let $z_1 = y_1$. Then z_1 satisfies (1). Assume now we have constructed $z_{n-1} = \sum_{i=q_{n-1}+1}^{q_n} \beta_i x_i$ with the required properties. Since $a = \{a_n\} \in c_0$, there exists a positive integer k such that $\sum_{i=k}^{k+q_n} a_i < 1/2^2$. Since $\{\alpha_n\}$ is decreasing to zero, choose h such that $p_{h+1} - p_h > k + q_n$ and

$\alpha_i^p < 1/2^2k$ for all i such that $p_h + 1 \leqq i \leqq p_{h+1}$. Define $q_{n+1} = p_{h+1} - p_h + q_n$, $\beta_{q_n+i} = \alpha_{p_h+i}$, $i = 1, 2, \cdots, q_{n+1} - q_n$; and $z_n = \sum_{i=q_n+1}^{q_{n+1}} \beta_i x_i$. Notice that the coefficients of z_n are the same as the coefficients of y_h; hence, $\| z_n \| = 1$. Now

$$\begin{aligned}
\sum_{i=q_n+1}^{q_{n+1}} \beta_i^p a_i &= \sum_{i=q_n+1}^{p_{h+1}-p_h+q} \alpha_{p_h-q_n+i}^p a_i \\
&= \sum_{i=1}^{p_{h+1}-p_h} \alpha_{p_h+i}^p a_i - \sum_{i=q_n+1}^{p_{h+1}-p_h+q_n} \alpha_{p_h-q_n+i}^p (a_{i-q} - a_i) \\
&= 1 - \sum_{i=q_n+1}^{q_n+k} \alpha_{p_h-q_n+i}^p (a_{i-q_n} - a_i) \\
&\quad - \sum_{i=q_n+k+1}^{p_{h+1}-p_h+q_n} \alpha_{p_h-q_n+i}^p (a_{i-q_n} - ai).
\end{aligned}$$

But

$$\begin{aligned}
\sum_{i=q_n+1}^{q_n+k} \alpha_{p_h-q_n+i}^p (a_{i-q_n} - a_i) &\leqq \sum_{i=q_n+1}^{q_n+k} \alpha_{p_h-q_n+i}^p \\
&< \frac{1}{2^2} \underbrace{\left(\frac{1}{k} + \cdots + \frac{1}{k} \right)}_{k \text{ times}} = \frac{1}{2^2};
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=q_n+k+1}^{p_h+1-p_h+q_n} \alpha_{p_h-q_n+i}^p (a_{i-q_n} - a_i) &\leqq \sum_{i=q_n+k+1}^{p_h+1-p_h+q_n} (a_{i-q_n} - a_i) \\
&= \sum_{i=1}^{q_n} a_{k+i} - \sum_{i=1}^{q_n} a_{p_{h+1}-p_h+i} \\
&\leqq \sum_{i=1}^{q_n} a_{k+i} < \frac{1}{2^2}.
\end{aligned}$$

Hence $\sum_{i=q_n+1}^{q_{n+1}} \beta_i^p a_i \geqq \frac{1}{2}$. By induction, we construct $\{z_n\}$ satisfying (1). Since $\{z_n\}$ is merely a translation of a subsequence of the block basic sequence $\{y_n\}$, it follows that $\{z_n\}$ is equivalent to a subsequence of $\{y_n\}$.

Finally, we claim that $\{z_n\}$ is equivalent to the unit vector basis of l^p. By Proposition 5, $\{z_n\}$ is a p-Hilbertian basic sequence. For any nonnegative scalars $b_1, b_2, \cdots, b_n$, we have

$$\left(\frac{1}{2}\right)^{1/p} \left(\sum_{i=1}^{n} b_i^p \right)^{1/p} \leqq \left[\sum_{i=1}^{n} b_i^p \left(\sum_{j=q_i+1}^{q_{i+1}} \beta_j^p a_j \right) \right]^{1/p} \leqq \left\| \sum_{i=1}^{n} b_i z_i \right\|.$$

Hence $\{z_n\}$ is a p-Besselian basic sequence. It follows that there is a subsequence of $\{y_n\}$ equivalent to the unit vector basis of the space l^p. Q.E.D.

COROLLARY 2. *Let $\{x_n\}$ be the unit vector basis of the Banach space $d(a,p)$. For every bounded block basic sequence $\{y_n\}$ of $\{x_n\}$, either there is a subsequence of $\{y_n\}$ which is equivalent to the unit vector basis of l^p or $\{y_n\}$ dominates $\{x_n\}$. In particular, every symmetric basic sequence in $d(a,p)$ dominates $\{x_n\}$.*

COROLLARY 3. *Let $\{x_n\}$ be the unit vector basis of $d(a,p)$. If $\{y_n\}$ is a bounded block basic sequence of $\{x_n\}$, then there is a block basic sequence of $\{y_n\}$ which is equivalent to the unit vector basis of l^p.*

PROOF. Let $y_n = \sum_{i=q_n+1}^{q_{n+1}} \alpha_i x_i$, $n = 1,2,\cdots$. Notice that $\inf_n \left\| \sum_{i=q_n+1}^{q_{n+1}} \alpha_i x_i \right\| > 0$ implies that $\sum_{i=1}^{\infty} \alpha_i x_i$ does not converge in $d(a,p)$. Since $\{x_n\}$ is a boundedly complete basis (see, e.q., [1]), it follows that $\sup_{k \leq n} \left\| \sum_{i=k}^{n} y_i \right\| = +\infty$. Therefore there exists a sequence $p_1 < p_2 < \cdots$ of integers such that $\sup_n \left\| \sum_{i=p_n+1}^{p_{n+1}} y_i \right\| = +\infty$. Let

$$z_n = \sum_{i=p_n+1}^{p_{n+1}} y_i \Big/ \left\| \sum_{i=p_n+1}^{p_{n+1}} y_i \right\|.$$

Considering $\{z_n\}$ as a bounded block basic sequence of $\{x_n\}$, it is easily seen that $\{z_n\}$ satisfies the hypotheses of Lemma 1. Hence, there is a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ which is equivalent to the unit vector basis of l^p. Q.E.D.

REMARK 1. If $\{y_n\}$ is a symmetric block basic sequence then it is known [e.g., 8] that there is a projection from $[\{y_n\}]$ onto $[\{z_{n_j}\}]$.

Let $\{x_n\}$ be the unit vector basis of $d(a,p)$. For any infinite-dimensional subspace X of $d(a,p)$, by a result of B. Bessaga and A. Pełczyński (see, e.g., [12, p. 442]), X contains a bounded basic sequence $\{y_n\}$ which is equivalent to a block basic sequence $\{z_n\}$ of $\{x_n\}$. By Corollary 3, the subspace $[\{z_n\}]$ contains a subspace which is isomorphic to l^p. Thus X contains a subspace Y which is isomorphic to l^p. In view of the previous remark, if X has a symmetric basis, then Y is complemented in X. Hence we obtain the following result.

THEOREM 1. *Every infinite dimensional subspace X of $d(a,p)$ contains a subspace Y which is isomorphic to l^p. If X has a symmetric basis then Y can be chosen to be complemented in X.*

REMARK 2. In [7, Proposition 4], it is proved that $d(a,p)$ has a complemented subspace isomorphic to l^p.

3. Uniqueness of symmetric basis in $d(a, p)$

Let $\{x_n\}$ be a symmetric basis of a Banach space X. If $\{x_n\}$ is not equivalent to the unit vector basis of c_0 or l^p, $1 \leqq p < +\infty$, then we know that there are bounded block basic sequences of $\{x_n\}$ which are not symmetric. On the other hand, if $\{y_n\}$ is a symmetric basic sequence in X, then either $\{y_n\}$ is equivalent to the unit vector basis of the space l^1 or $\{y_n\}$ is weakly convergent to zero. In the latter case, $\{y_n\}$ is equivalent to a bounded block basic sequence of $\{x_n\}$. In this section, we shall construct some special symmetric basic sequences in X and in the $d(a,p)$ spaces we will determine all the bounded block basic sequences of the unit vector basis which are symmetric. A new type of block basic sequence is introduced which seems to play an important role in determining symmetric basic sequences in Banach spaces with symmetric bases.

PROPOSITION 6. *Let $\{x_n\}$ be a symmetric basis in a Banach space X. For any $\{a_n\} \in l^1$, $a_1 \neq 0$, and any monotone increasing sequence of natural numbers, $p_1 < p_2 < \cdots < p_n < \cdots$, let $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_{i-p_n} x_i$ for $n = 1,2,\cdots$. Then $\{y_n\}$ is a basic sequence in X which is equivalent to the basis $\{x_n\}$.*

PROOF. By Proposition 4, it is clear that the basic sequence $\{y_n\}$ dominates the basis $\{x_n\}$. Conversely, let $\sum_{n=1}^{\infty} \alpha_n x_n \in X$. Then

$$\left\| \sum_{i=1}^{n} \alpha_i y_i \right\| \leqq \left(\sum_{i=1}^{\infty} |a_i| \right) \left\| \sum_{i=1}^{p_n+1} \alpha_i x_i \right\|, \qquad n = 1,2,\cdots.$$

Thus $\sum_{i=1}^{n} \alpha_i y_i$ converges in X. Hence $\{y_n\}$ is equivalent to $\{x_n\}$. Q.E.D.

THEOREM 2. *Let $\{x_n\}$ be a symmetric basis of a Banach space X. For any element $\sum_{n=1}^{\infty} \alpha_n x_n \in X$, $\alpha_1 \neq 0$, and for any natural numbers $p_1 < p_2 < \cdots$, let $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_{i-p_n} x_i$ for $n = 1,2,\cdots$. Then there is a subsequence of $\{y_n\}$ which is a symmetric basic sequence in X.*

PROOF. If $\sup_{1 \leqq n < +\infty} (p_{n+1} - p_n) < +\infty$, then $\{y_n\}$ is equivalent to the basis $\{x_n\}$ and we are done. Assume that $\sup_{1 \leqq n < +\infty} (p_{n+1} - p_n) = +\infty$. By switching to a subsequence, if necessary, we may assume that $p_{n+2} - p_{n+1} > p_{n+1} - p_n$ for $n = 1,2,\cdots$.

Let $\{N_i\}_{i=1,2,\dots}$ be subsets of the natural numbers, N, such that $N = \bigcup_{i=1}^{\infty} N_i$, $N_i \cap N_j = \varnothing$ for all $i \neq j$ and $\bar{\bar{N}}_i = \bar{\bar{N}}$, $i = 1,2,\cdots$. For each i, $N_i = \{i,j\}_{j=1,2,\dots}$, let $u_i = \sum_{j=1}^{\infty} \alpha_j x_{i,j}$. It is clear that $\{u_i\}$ is a symmetric basic sequence in $d(a,p)$.

Let $\{n_i\}$ be an increasing sequence such that $\left\| \sum_{j=p_{n_i+1}-p_{n_i}+1}^{\infty} \alpha_j x_j \right\| < 1/2^i$, $i = 1,2,\cdots$. Let $z_i = \sum_{j=1}^{p_{n_i+1}-p_{n_i}} \alpha_j x_{i,j}$, $i = 1,2,\cdots$. Then

$$\sum_{i=1}^{\infty} \| u_i - z_i \| = \sum_{i=1}^{\infty} \Big\| \sum_{j=p_{n_{i+1}}-p_{n_i}+1}^{\infty} \alpha_j x_{i\,j} \Big\| < \sum_{i=1}^{\infty} 1/2^i = 1.$$

By a theorem of B. Bessaga and A. Pełczyński (e.g. [12, p. 93]), $\{u_i\} \sim \{z_i\}$. Now, it is clear that $\{z_i\} \sim \{y_{n_i}\}$. Hence, $\{u_i\} \sim \{y_{n_i}\}$ and so $\{y_{n_i}\}$ is symmetric. Q.E.D.

Remark 3. For $1 \leqq p < +\infty$, the symmetric basic sequences in the Lorentz sequence space $d(a,p)$ constructed in Theorem 2 are not equivalent to the unit vector basis of l^p. Indeed, if $0 \neq \sum_{n=1}^{\infty} \alpha_n x_n \in d(a,p)$ where $\{x_n\}$ is the unit vector basis of $d(a,p)$, we may assume that $1 \geqq \alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n \geqq \cdots \geqq 0$. Since $\lim_{i\to\infty} \| \sum_{n=1}^{\infty} \alpha_{i+n} x_n \| = 0$, for any $\varepsilon > 0$, there exists a positive integer N_ε such that $\| \sum_{n=1}^{\infty} \alpha_{N_\varepsilon+n} x_n \|^p < \varepsilon^p/2$. Let $a = \{a_n\}$. Since $\lim_{n\to\infty} a_n = 0$, we have that $\lim_{n\to\infty} \sum_{i=1}^{nN_\varepsilon} a_i/_n = 0$. Choose n such that $\sum_{i=1}^{nN_\varepsilon} a_i/n < \varepsilon^p/2$. Then if $y_j = \sum_{i=p_j+1}^{p_{j+1}} \alpha_{i-p_j} x_i$ for $j = 1,2,\cdots$,

$$\Big\| \sum_{i=1}^{n} y_{N_\varepsilon+i} \Big\|^p \leqq \sum_{i=1}^{N_\varepsilon} \alpha_i^p \Big(\sum_{k=(i-1)n+1}^{in} a_k \Big) + n \Big\| \sum_{i=1}^{\infty} \alpha_{N_\varepsilon+i} x_i \Big\|^p$$

$$< \sum_{i=1}^{nN_\varepsilon} a_i + n\Big(\frac{\varepsilon^p}{2}\Big) < n\Big(\frac{\varepsilon^p}{2}\Big) + n\Big(\frac{\varepsilon^p}{2}\Big) = n\varepsilon^p.$$

Hence $\| \sum_{i=1}^{n} y_{N_\varepsilon+i} \| < n^{1/p}\varepsilon$. But $n^{1/p} = \| \sum_{i=1}^{n} e_i \|$ where $\{e_n\}$ is the unit vector basis of l^p. Thus $\{y_n\}$ is not equivalent to $\{e_n\}$. Similarly, no subsequence of $\{y_n\}$ is equivalent to $\{e_n\}$.

Definition. Let $\{x_n\}$ be a symmetric basis of a Banach space X. For any $\sum_{n=1}^{\infty} \alpha_n x_n \in X$, $\alpha_1 \neq 0$, and any $p_1 < p_2 < \cdots < p_n < \cdots$, let $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_{i-p_n} x_i$ for $n = 1,2,\cdots$. Then $\{y_n\}$ is a bounded block basic sequence of $\{x_n\}$ in X. We shall call $\{y_n\}$ a block of type I of $\{x_n\}$.

Theorem 3. *Let $\{x_n\}$ be the unit vector basis of the Lorentz sequence space $d(a,p)$. For any bounded block basic sequence $\{y_n\}$ of $\{x_n\}$, $\{y_n\}$ has a subsequence equivalent either to the unit vector basis of l^p or to a block basic sequence of type I of $\{x_n\}$.*

Proof. Let $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$ for $n = 1,2,\cdots$. We may assume that $\| y_n \| = 1$ and $\alpha_{p_n+1} \geqq \alpha_{p_n+2} \geqq \cdots \geqq \alpha_{p_{n+1}} > 0$ for $n = 1,2,\cdots$. If $\sup_{1\leqq n<+\infty}(p_{n+1} - p_n) < +\infty$, then $\{y_n\}$ is equivalent to $\{x_n\}$ and so is equivalent to a block of type I of $\{x_n\}$. Assume now that $\sup_{1\leqq n<+\infty}(p_{n+1} - p_n) = +\infty$.

Let $\beta_i = \sup_{1 \leqq n < +\infty} |\alpha_{p_n+i}|$, $i = 1,2,\cdots$. We first observe that in $d(a,p)$, for every $\varepsilon > 0$, there exists $n(\varepsilon)$ such that $\|(\varepsilon,\varepsilon,\cdots,\varepsilon,0,0,\cdots)\| > 1$ where the number of ε's is $n(\varepsilon)$. Thus $\lim_{i\to\infty}\beta_i = 0$.

Case 1. Assume that for every $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that $\|\sum_{i=p_n+N}^{p_{n+1}} \alpha_i x_i\| \leqq \varepsilon$ for all n with $p_{n+1} - p_n \geqq N$. Since $\sup_{1\leqq n<+\infty}(p_{n+1} - p_n) = +\infty$, by switching to a subsequence of $\{y_n\}$ if necessary, we may assume that $p_{n+2} - p_{n+1} \geqq p_{n+1} - p_n$, $n = 1,2,\cdots$. For each $n = 1,2,\cdots$, define

$$z_n = \sum_{i=1}^{p_{n+1}-p_n} \alpha_{i+p_n} x_i.$$

Then $\|z_n\| = \|y_n\| = 1$ for $n = 1,2,\cdots$. By hypothesis and by using a standard argument of compactness, it can be shown that there is a Cauchy subsequence of $\{z_n\}$. Thus we may assume that $\lim_{n\to\infty} z_n = x = \sum_{i=1}^{\infty}\beta_i x_i \in d(a,p)$. Since $\|z_n\| = \|y_n\| = 1$, it is clear that $x \neq 0$.

Let $\{f_n\}$ be a sequence of biorthogonal functionals of $\{y_n\}$. Then $\sup_{1<n<+\infty}\|f_n\| = M < +\infty$. Since $\lim_{n\to\infty} z_n = x$, there is a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $\sum_{i=1}^{\infty}\|z_{n_i} - x\| < 1/M$. Define

$$u_i = \sum_{k=p_{n_i}+1}^{p_{n_i+1}} \beta_{k-p_{n_i}} x_k, \qquad i = 1,2,\cdots.$$

Then $\{u_i\}$ is a block of type I of $\{x_n\}$ and

$$\sum_{i=1}^{\infty} \|f_{n_i}\| \, \|y_{n_i} - u_i\| \leqq \sum_{i=1}^{\infty} \|f_{n_i}\| \, \|z_{n_i} - x\| < 1.$$

Hence, $\{u_i\} \sim \{y_{n_i}\}$.

Case 2. There exists an $\varepsilon > 0$ such that for every $N = 1,2,\cdots$, there exists $n(N)$ such that $p_{n+1} - p_n \geqq N$ and $\|\sum_{i=p_n+N}^{p_{n+1}}\alpha_i x_i\| > \varepsilon$. Hence there exists $n_1 < n_2 < \cdots$ such that $p_{n_i+1} - p_{n_i} > i$ and $\|\sum_{j=p_{n_i}+i}^{p_{n_i+1}}\alpha_j x_j\| > \varepsilon$. Since $\lim_{i\to\infty}\sup_n|\alpha_{p_n+i}| = 0$, we may assume that α_j is monotone decreasing to zero. For each $i = 1,2,\cdots$, let $z_i = \sum_{j=p_{n_i}+i}^{p_{n_i+1}}\alpha_j x_j$. Then $\varepsilon \leqq \|z_i\| \leqq \|y_{n_i}\| = 1$ for $i = 1,2,\cdots$. Hence $\{z_i\}$ is a bounded block basic sequence of $\{x_n\}$ and the coefficients of $\{z_i\}$ tend to zero. By Lemma 1, there is a subsequence, say $\{w_i\}$, of $\{z_i\}$ which is equivalent to the unit vector basis $\{e_n\}$ of l^p. Since $\{y_{n_i}\}$ dominates $\{w_i\}$, $\{y_{n_i}\}$ is a p-Besselian basic sequence. By Proposition 5, the basic sequence $\{y_{n_i}\}$ is p-Hilbertian. Therefore, $\{y_{n_i}\}$ is equivalent to $\{e_n\}$. Q.E.D.

COROLLARY 4. *Let $\{x_n\}$ be the unit vector basis of the Banach space $d(a,p)$.*

Then every bounded block basic sequence of $\{x_n\}$ has a subsequence which is symmetric.

COROLLARY 5. *Every symmetric basic sequence in the space $d(a,p)$ is equivalent either to the unit vector basis of l^p, or to a block basic sequence of type I of the unit vector basis of $d(a,p)$.*

THEOREM 4. *If Y is a closed linear subspace of $d(a,p)$ with symmetric basis, then all symmetric bases in Y are equivalent.*

PROOF. Let $\{y_n\}$ be a symmetric basis of Y. By Corollary 5, $\{y_n\}$ is equivalent either to the unit vector basis $\{e_n\}$ of l^p or to a block basic sequence of type I in $d(a,p)$. In the first case, it is clear that all symmetric bases in Y are equivalent. Otherwise, we may assume that $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$, $n = 1,2,\cdots$, such that $\mathrm{m}_{n\to\infty} \alpha_n \neq 0$ where $\{x_n\}$ is the unit vector basis of $d(a,p)$. Let $z_n = \sum_{i=q_n+1}^{q_{n+1}} \beta_i y_i$, $n = 1,2,\cdots$ be another symmetric basis of Y. Since $\{z_n\}$ and $\{e_n\}$ are not equivalent, $\lim_{n\to\infty} \beta_n \neq 0$. By Proposition 4 and the symmetricity of $\{z_n\}$, we have $\{z_n\} > \{y_n\}$. By the same argument, $\{y_n\} > \{z_n\}$. Q.E.D.

4. $d(a,p)$ with exactly two nonequivalent symmetric basic sequences

In this section, we give a necessary and sufficient condition that $d(a,p)$ has exactly two nonequivalent symmetric basic sequences.

DEFINITION. Let $\{s_n\}$ and $\{t_n\}$ be two sequences of nonnegative numbers. We say that $\{t_n\}$ dominates $\{s_n\}$, denoted by $t_n > s_n$. if there exists a positive number A such that $s_n \leqq At_n$, $n = 1,2,\cdots$. We say that $\{s_n\}$ is equivalent to $\{t_n\}$, and write $s_n \sim t_n$, if $s_n > t_n$ and $t_n > s_n$.

PROPOSITION 7. *Let $\{v_i\}$ and $\{w_i\}$ be sequences of nonnegative numbers and let $s_n = \sum_{i=1}^{n} v_i, t_n = \sum_{i=1}^{n} w_i, n = 1,2,\cdots$. Then $t_n > s_n$ if and only if there exists $A > 0$ such that $\sum_{i=1}^{\infty} \beta_i v_i \leqq A \sum_{i=1}^{\infty} \beta_i w_i$ for all nonincreasing sequences $\{\beta_i\}$ of nonnegative numbers.*

The proof is obvious.

LEMMA 2. *Let $d(a,p)$ and $d(b,p)$ be Lorentz sequence spaces. For each $n = 1,2,\cdots$, let $s_n = \sum_{i=1}^{n} a_i, t_n = \sum_{i=1}^{n} b_i$ where $a = (a_1, a_2, \cdots)$ and $b=(b_1, b_2, \cdots)$. Then $d(a,p)$ is isomorphic to $d(b,p)$ if and only if $s_n \sim t_n$.*

PROOF. Let $\{x_n\}$ and $\{y_n\}$ be the unit vector basis of $d(a,p)$ and $d(b,p)$, respectively. By Proposition 7, $s_n \sim t_n$ if and only if there exist $A > 0$, $B > 0$ such that $A\sum_{n=1}^{\infty} \alpha_n^p b_n \leqq \sum_{n=1}^{\infty} \alpha_n^p a_n \leqq B\sum_{n=1}^{\infty} \alpha_n^p b_n$ for every nonincreasing sequence $\{\alpha_n\}$. Since $\{x_n\}$ and $\{y_n\}$ are symmetric, this means that $s_n \sim t_n$ if and only if $\{x_n\} \sim \{y_n\}$. Finally, notice that $\{x_n\}$ and $\{y_n\}$ are unique, up to equivalence, symmetric bases in $d(a,p)$ and $d(b,p)$ respectively. Then $d(a,p)$ is isomorphic to $d(b,p)$ if and only if $\{x_n\} \sim \{y_n\}$, i.e., if and only if $s_n \sim t_n$. Q.E.D.

PROPOSITION 8. *Let $d(a,p)$ be a Lorentz sequence space. For $\alpha = \sum_{n=1}^{\infty} \alpha_n x_n \in d(a,p)$, let $v_i = \sum_{j=1}^{\infty} \alpha_j^p a_{i,j}$ where $\{a_{i,j}\}_{j=1,2\,\cdots}$ (respectively, $\{a_{i,j}\}_{i=1\,2\,\cdots}$) is a subsequence of $\{a_n\}$ for $i = 1,2,\cdots$ (respectively, $j = 1,2,\cdots$). Then $\{v_i\}$ is decreasing to zero.*

PROOF. Notice that from the hypothesis, $a_{i,j} \leqq a_{i,h}$ $a_{j,i} \leqq a_{h,i}$ for $h \leqq j$, and $a_{i,j} \leqq a_i$, $i,j = 1,2,\cdots$. Therefore, $\{v_i\}$ is decreasing. For any $\varepsilon > 0$, choose N and M such that $\sum_{j=N+1}^{\infty} \alpha_j^p a_j < \varepsilon/2$ and $a_M < \varepsilon/2\sum_{j=1}^{N} \alpha_j^p$. Then for every $i \geqq M$, $v_i = \sum_{j=1}^{N} \alpha_j^p a_{i,j} + \sum_{j=N+1}^{\infty} \alpha_j^p a_{i,j} \leqq a_i \sum_{j=1}^{N} \alpha_j^p + \sum_{j=N+1}^{\infty} \alpha_j^p a_j < \varepsilon$. Q.E.D.

DEFINITION. Let $0 \neq \alpha = \sum_{n=1}^{\infty} \alpha_n x_n \in d(a,p)$ with $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$. For each $n = 1,2,\cdots$, let $s_n = \sum_{i=1}^{n} a_i$, $s_n^{(\alpha)} = \sum_{i=1}^{\infty} \alpha_i^p (s_{ni} - s_{n(i-1)})$ where $s_0 = 0$, $w_n^{(\alpha)} = s_n^{(\alpha)} - s_{n-1}^{(\alpha)}$ and $s_0^{(\alpha)} = 0$.

PROPOSITION 9. *For every $\alpha \in d(a,p)$, $s_n^{(\alpha)} \leqq n\|\alpha\|^p$, $\sum_{n=1}^{\infty} w_n^{(\alpha)} = +\infty$ and $\{w_n^{(\alpha)}\}$ is a sequence decreasing to zero.*

PROOF. Clearly $s_n^{(\alpha)} \leqq n\|\alpha\|^p$ and $\sum_{n=1}^{\infty} w_n^{(\alpha)} = +\infty$. For a fixed n, we have $s_{(n+1)k} \geqq s_{nk}$ and $2s_{nk} \geqq s_{(n+1)k} + s_{(n-1)k}$ for $k = 1,2,\cdots$. Thus $\sum_{i=1}^{k} (s_{(n+1)i} - s_{(n+1)(i-1)}) = s_{(n+1)k} \geqq s_{nk} = \sum_{i=1}^{k} (s_{ni} - s_{n(i-1)})$ and by Proposition 7 with $\beta_i = \alpha_i$, we get $s_{n+1}^{(\alpha)} \geqq s_n^{(\alpha)}$ and $2s_n^{(\alpha)} \geqq s_{n+1}^{(\alpha)} + s_{n-1}^{(\alpha)}$. Hence $\{w_n^{(\alpha)}\}$ is a decreasing sequence of nonnegative numbers. Now,

$$w_n^{(\alpha)} = \sum_{i=1}^{\infty} \alpha_i^p \left(\sum_{j=1}^{n} a_{n(i-1)+j} - \sum_{j=1}^{n-1} a_{(n-1)(i-1)+j} \right) \leqq \sum_{i=1}^{\infty} \alpha_i^p a_{ni}.$$

Thus $\lim_{n\to\infty} w_n^{(\alpha)} = 0$ follows from Proposition 8. Q.E.D.

LEMMA. 3. *If for every $\alpha \in d(a,p)$ with $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$ and $\|\alpha\| = 1$, there exists $B_\alpha > 0$ such that $s_n^{(\alpha)} \leqq B_\alpha s_n$, $n = 1,2,\cdots$, then there exists $B > 0$ such that for all α, $\|\alpha\| = 1$, in $d(a,p)$, $s_n^{(\alpha)} \leqq Bs_n$, $n = 1,2,\cdots$.*

PROOF. For every fixed n, let $a_k^{(n)} = (s_{nk} - s_{n(k-1)})/s_n$, $k = 1,2,3,\cdots$. Then $a^{(n)} = (a_1^{(n)}, a_2^{(n)}, \cdots, a_k^{(n)}, \cdots) \in c_0 \setminus l^1$. Let $d = (\sum_{n=1}^{\infty} \oplus d(a^{(n)},p))_{c_0}$ and let

$\{x_i^{(n)}\}_{i=1,2,\cdots}$ and $\{x_i\}$ be the unit vector basis of $d(a^{(n)},p)$ and $d(a,p)$, respectively. Define T_n: $d(a,p) \to d$ by

$$T_n\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) = \Big(0,\cdots,0,\underbrace{\sum_{i=1}^{\infty} \alpha_i x_i^{(n)}}_{n\text{'th place}},0,\cdots\Big)$$

for all $\alpha = \sum_{i=1}^{\infty} \alpha_i x_i \in d(a,p)$. Then if $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \cdots \geqq 0$, we have

$$\| T_n(\alpha) \|^p = \sum_{i=1}^{\infty} \frac{\alpha_i^p(s_{ni} - s_{n(i-1)})}{s_n} = \frac{s_n^{(\alpha)}}{s_n} \leqq \frac{n}{s_n} \| \alpha \|^p$$

and so $\| T_n \| \leqq (n/s_n)^{1/p}$. Now for each $\alpha \in d(a,p)$, by the hypothesis, $\| T_n(\alpha) \| \leqq B_\alpha$ for all $n = 1,2,\cdots$. By the uniform boundedness principle, there exists $B > 0$ such that $\sup_n \| T_n \| < B^{1/p}$. Thus for every $\alpha \in d(a,p)$, $\| \alpha \| = 1$, we get $s_n^{(\alpha)} \leqq Bs_n$, $n = 1,2,\cdots$. Q.E.D.

THEOREM 5. *Let $d(a,p)$ be a Lorentz sequence space. Then* $\sup_{1 \leqq n,k < +\infty} s_{nk}/s_n s_k < +\infty$ *if and only if for every $\alpha \in d(a,p)$, $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$, $\| \alpha \| = 1$, $s_n^{(\alpha)} \sim s_n$.*

PROOF. Let $\alpha = \sum_{n=1}^{\infty} \alpha_n x_n \in d(a,p)$ such that $\| \alpha \| = 1$. Clearly we always have $s_n^{(\alpha)} \geqq \alpha_1^p s_n$. Suppose $\sup_{1 \leqq n,k < +\infty} s_{nk}/s_n s_k = B < +\infty$. Then $s_{nk} = \sum_{i=1}(s_{ni} - s_{n(i-1)}) \leqq Bs_n(\sum_{i=1}^k a_i) = Bs_n s_k$ for all $n,k = 1,2,\cdots$. Fix n. By Proposition 7, we get $Bs_n = Bs_n(\sum_{i=1}^{\infty} \alpha_i^p a_i) \geqq \sum_{i=1}^{\infty} \alpha_i^p(s_{ni} - s_{n(i-1)}) = s_n^{(\alpha)}$. Hence $s_n^{(\alpha)} \sim s_n$.

Conversely, suppose $s_n^{(\alpha)} \sim s_n$ for all $\alpha \in d(a,p)$, $\| \alpha \| = 1$ and $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq 0$. By Lemma 3, there exists $B > 0$ such that for all $\| \alpha \| = 1$, $s_n^{(\alpha)} \leqq Bs_n$, $n = 1,2,\cdots$. For each k, let $\gamma_i = (1/s_k)^{1/p}$ if $i \leqq k$ and $\gamma_i = 0$ if $i > k$. Let $\gamma = \sum_{i=1}^{\infty} \gamma_i x_i$. Then $\| \gamma \| = 1$ and $s_n^{(\gamma)} = s_{nk}/s_k$. Hence $s_{nk} \leqq Bs_n s_k$, $n,k = 1,2,\cdots$. This completes the proof of the theorem. Q.E.D.

LEMMA 4. *Let $\alpha = \sum_{n=1}^{\infty} \alpha_n x_n \in d(a,p)$ such that $\| \alpha \| = 1$ and $\alpha_1 \geqq \alpha_2 \geqq \cdots$, $\geqq 0$. If the block basic sequence $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_{i-p_n} x_i$, $n = 1,2,\cdots$ is symmetric then $[\{y_n\}]$ is isomorphic to $d(a,p)$ if and only if $s_n^{(\alpha)} \sim s_n$.*

PROOF. Let $\{N_i\}_{i=1,2,\cdots}$ be subsets of the natural numbers, N, such that $N = \bigcup_{i=1}^{\infty} N_i$, $N_i \cap N_j = \emptyset$ for all $i \neq j$, and $\bar{\bar{N_i}} = \bar{\bar{N}}$, $i = 1,2,\cdots$. For each i, $N_i = \{i,j\}_{j=1,2,\cdots}$. Let $u_i^{(\alpha)} = \sum_{j=1}^{\infty} \alpha_j x_{i,j}$ where $\alpha = \sum_{j=1}^{\infty} \alpha_j x_j \in d(a,p)$. As we have seen in the proof of Theorem 2, $\{y_n\}$ is equivalent to $\{u_n^{(\alpha)}\}$ and $\| \sum_{i=1}^n u_i^{(\alpha)} \|^p = s_n^{(\alpha)}$. Suppose that $[\{y_n\}]$ is isomorphic to $d(a,p)$. Then $[\{u_n^{(\alpha)}\}]$ is isomorphic to $d(a,p)$, and since all symmetric bases in $d(a,p)$ are equivalent, $\{u_n^{(\alpha)}\}$ is equivalent to $\{x_n\}$. Thus $\| \sum_{i=1}^n u_i^{(\alpha)} \| \sim \| \sum_{i=1}^n x_i \|$ which means $s_n^{(\alpha)} \sim s_n$.

Conversely, suppose $s_n^{(\alpha)} \sim s_n$. Let $w_1^{(\alpha)} = s_1^{(\alpha)}$, $w_n^{(\alpha)} = s_{n+1}^{(\alpha)} - s_n^{(\alpha)}$, $n = 2,3,\cdots$ and $w^{(\alpha)} = (w_1^{(\alpha)}, w_2^{(\alpha)}, \cdots)$. By Lemma 2, $d(w^{(\alpha)}, p)$ is isomorphic to $d(a,p)$. Let $\{\beta_n\}$ be any decreasing sequence of nonnegative numbers. Then

$$\left\| \sum_{i=1}^{N} \beta_i u_i^{(\alpha)} \right\|^p = \sum_{i=1}^{N} \beta_i^p \left(\sum_{j=1}^{\infty} \alpha_i^p a_{i,j} \right)$$

where for every $i = 1,2,\cdots,N$ (respectively, for every j), $\{a_{i,j}\}_{j=1,2,\dots}$ is a decreasing subsequence of $\{a_n\}$. Now, for every l and k,

$$\sum_{i=1}^{k} \left(\sum_{j=1}^{l} a_{i,j} \right) \leqq s_{kl} = \sum_{j=1}^{l} (s_{kj} - s_{k(j-1)}).$$

For each fixed $k = 1,2,\cdots,N$, by Proposition 7

$$\sum_{i=1}^{k} \left(\sum_{j=1}^{\infty} \alpha_j^p a_{i,j} \right) \leqq \sum_{j=1}^{\infty} \alpha_j^p (s_{kj} - s_{k(j-1)}) = s_k^{(\alpha)}.$$

Since $\{\beta_n\}$ is decreasing, by Proposition 7 again, $\| \sum_{i=1}^N \beta_i u_i^{(\alpha)} \|^p \leqq \sum_{i=1}^N \beta_i^p w_i^{(\alpha)}$. Hence $\{v_n^{(\alpha)}\} > \{u_n^{(\alpha)}\}$ where $\{v_n^{(\alpha)}\}$ is the unit vector basis of $d(w^{(\alpha)}, p)$. Since $\{x_n\} \sim \{v_n^{(\alpha)}\}$ and $\{u_n^{(\alpha)}\} \sim \{y_n\}$ we get $\{x_n\} > \{y_n\}$. On the other hand, by Proposition 4, $\{y_n\} > \{x_n\}$. Thus $[\{y_n\}]$ is isomorphic to $d(a,p)$. Q.E.D.

THEOREM 6. *In $d(a,p)$ there are exactly two nonequivalent symmetric basic sequences if and only if* $\sup_{1 \leqq n\, k < +\infty} s_{nk}/s_n s_k < +\infty$.

PROOF. Let $\{y_n\}$ be a symmetric basic sequence in $d(a,p)$. By proposition 1 and Theorem 3, $\{y_n\}$ is equivalent either to the unit vector basis of l^p or to a block basic sequence of type I. If $\sup_{1 \leqq n,k < +\infty} s_{nk}/s_n s_k < +\infty$, by Theorem 5 and Lemma 4, $\{y_n\}$ is equivalent to the unit vector basis $\{x_n\}$ of $d(a,p)$. Conversely, if $\sup_{1 \leqq n\, k < +\infty} s_{nk}/s_n s_k = +\infty$, by Theorem 5 and Lemma 4, there exists a block basic sequence $\{y_n\}$ of Type I which is not equivalent to $\{x_n\}$. By Remark 3, $\{y_n\}$ is not equivalent to unit vector basis of l^p. Thus, in $d(a,p)$ there are more than two nonequivalent symmetric basic sequences. Q.E.D.

Let us remark that there exists a Lorentz sequence space with infinitely many nonequivalent symmetric basic sequences. Indeed, it has been mentioned in [7, p. 378] that the Lorentz sequence space $d(\{1/\log n\}, p)$ is isomorphic to the Orlicz sequence space l_M where $M(x) = x^p/1 + |\log x|$; furthermore, in the same paper [7, p. 363] it has been proved that l_M has infinitely many nonequivalent symmetric (Orlicz) basic sequences.

THEOREM 7. *There exists a Lorentz sequence space $d(a,p)$ having a subspace* *with symmetric basis which is isomorphic neither to l^p nor to any Lorentz sequence space.*

PROOF. Let $p \geqq 1$ and consider the Lorentz sequence space $d(a,p)$ for which $a_1 = a_2 = 1$, $a_n = 1/\sqrt{n}(\log n)^2$, $n = 3,4,\cdots$. Let $\alpha_n = n^{-\frac{1}{2}p}$, $n = 1,2,\cdots$. Then $\alpha = \{\alpha_n\} \in d(a,p)$. Define the vectors $\{u_i^{(\alpha)}\}$ as in the proof of Lemma 4. One can easily see that if $[\{u_i^{(\alpha)}\}]$ is isomorphic to a Lorentz sequence space, then $\{u_i^{(\alpha)}\}$ is equivalent to the unit vector basis of $d(w^{(\alpha)},p)$. But by definition,

$$s_n^{(\alpha)} = \sum_{i=1}^{\infty} \alpha_i^p (s_{ni} - s_{n(i-1)}) \geqq n \sum_{i=1}^{\infty} \alpha_i^p a_{ni}, \qquad n = 1,2,\cdots,$$

and

$$n \sum_{i=1}^{n} \alpha_i^p a_{ni} \sim n \int_1^{\infty} \frac{dx}{x\sqrt{n}(\log nx)^2} = \frac{\sqrt{n}}{\log n}.$$

Consequently,

$$\sum_{j=1}^{n} \frac{w_i^{(\alpha)}}{\sqrt{j}} \geqq \sum_{j=1}^{n} \left(\frac{1}{\sqrt{j}} - \frac{1}{\sqrt{j+1}}\right) s_j^{(\alpha)} \geqq \sum_{j=1}^{n} \left(\frac{1}{\sqrt{j}} - \frac{1}{\sqrt{j+1}}\right) j \sum_{i=1}^{\infty} \alpha_j^p a_{ij}$$

$$\sim \sum_{j=1}^{n} \left(\frac{1}{\sqrt{j}} - \frac{1}{\sqrt{j+1}}\right) \frac{\sqrt{j}}{\log j} \sim \sum_{j=1}^{n} \frac{1}{(j+1)\log(j+1)}.$$

On the other hand

$$\left\| \sum_{i=1}^{\infty} i^{-1/(2p)} u_i^{(\alpha)} \right\|^p \leqq 1 + \sum_{n=2}^{\infty} \frac{d(n)}{\sqrt{n}} a_{\sum_{i=1}^{n-1} (d i)}$$

where $d(n)$ is the number of divisors of n. Since $\sum_{i=1}^{n} d(i) \sim n \log n$ [5, p. 262] there exists a constant $A > 0$ such that

$$\sum_{n=2}^{\infty} \frac{d(n)}{\sqrt{n}} a_{\sum_{i=1}^{n-1} (d i)} \leqq A \sum_{n=2}^{\infty} \frac{d(n)}{n(\log n)^{5/2}},$$

$$= A \sum_{n=2}^{\infty} \left[\frac{1}{n(\log n)^{5/2}} - \frac{1}{(n+1)(\log(n+1))^{5/2}}\right] \sum_{i=2}^{n} d(i)$$

$$\sim \sum_{n=2}^{\infty} \frac{(\log n)^{5/2}}{n^2(\log n)^5} n \log n = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^{3/2}} < +\infty.$$

Hence $\sum_{i=1}^{\infty} i^{-1/2p} u_i^{(\alpha)}$ converges while the sequence $\{i^{-1/(2p)}\} \notin d(w^{(\alpha)},p)$. This means that $[\{u_i^{(\alpha)}\}]$ is isomorphic to no Lorentz sequence space. To conclude the proof, notice that $[\{u_i^{(\alpha)}\}]$ is not isomorphic to l^p (cf. Remark 3). Q.E.D.

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